Optimal Portfolio Selection: Mean-Variance versus Mean-VaR

ABSTRACT

This paper proposes a new approach to optimal portfolio selection in a value-at-risk (VaR) framework. A mean-VaR approach is introduced to allocate financial assets by maximizing the expected value of some utility function approximated by the expected return and VaR of the portfolio as well as the investor’s aversion to value-at-risk. The relative performance of the mean-variance and mean-VaR portfolio selection models is compared in terms of the optimal portfolios’ return as well as the expected risk premium per unit of risk. The empirical findings indicate that the new approach generally performs better than the mean-variance model over the entire sample period (1985-2000) and across alternative investment horizons. In addition, the paper provides a new and robust procedure to estimating time-varying risk tolerance in a dynamic asset allocation framework, and shows that the investor’s aversion to portfolio risk depends on the expected returns, variance-covariance matrix, and the set of portfolio weights. The results indicate that the risk tolerance in mean-variance and mean-VaR framework is time-varying and closely related to the standard deviation of the portfolio.
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I. Introduction

The modern theory of portfolio choice determines the optimum asset mix by maximizing (1) the expected risk premium per unit of risk in a mean-variance framework or (2) the expected value of some utility function approximated by the expected return and variance of the portfolio. In both cases, market risk of the portfolio is defined in terms of the variance (or standard deviation) of expected portfolio returns. Modeling portfolio risk with the traditional volatility measures implies that investors are concerned only about the average variation (and co-variation) of individual stock returns, and they are not allowed to treat the negative and positive tails of the return distribution separately. The standard risk measures determine the volatility of unexpected outcomes under normal market conditions, which corresponds to the normal functioning of financial markets during ordinary periods. However, neither the variance nor the standard deviation can yield an accurate characterization of actual portfolio risk during highly volatile periods. Therefore, the set of mean-variance efficient portfolios may produce an inefficient strategy for maximizing expected return of the portfolio while minimizing its risk.

This paper proposes a new approach to optimal portfolio selection in a value-at-risk (VaR) framework. A mean-VaR approach is introduced to allocate financial assets by maximizing the expected value of some utility function approximated by the expected return and VaR of the portfolio as well as the investor’s aversion to value-at-risk. The focus on VaR as the appropriate measure of portfolio risk allows investors to treat losses and gains asymmetrically.

Another contribution of the paper is to provide a new approach to estimating the investor’s risk aversion parameter. A robust procedure is introduced to calculate time-varying risk tolerance in the mean-variance and mean-VaR asset allocation models. As discussed in Chopra and Ziemba (1993), expected utility maximization in a mean-variance framework is very sensitive to errors in the estimates of the inputs. They show that the relative impact of errors in means, variances, and covariances depends on the investor’s risk tolerance. However, as indicated by Campbell, Huisman, and Koedijk (2001), the existing literature on expected utility theory is limited as to measuring the degree of risk aversion. The earlier research does not provide any methods that can be used to determine the magnitude of investors’ risk tolerance. This study shows that the investor’s aversion to risk depends on the expected returns, variances, covariances, and the set of portfolio weights. The empirical findings indicate that the risk tolerance in mean-variance and mean-VaR framework is time-varying and closely related to the standard deviation of the portfolio.
The relative performance of the mean-variance and mean-VaR portfolio selection models is compared in a dynamic asset allocation framework. An investor’s dynamic asset allocation decision is based on a risk-return trade-off. The objective of the dynamic asset allocator is to find the optimum asset mix by maximizing the expected utility of his wealth at each period in time. The investor’s expected utility is defined as a function of the expected return of his portfolio minus the penalty for risk (e.g., risk tolerance times VaR of the portfolio). In a mean-variance framework, market risk of the portfolio depends only on the variance-covariance matrix and the risk tolerance. However, in a mean-VaR framework, in addition to the variance-covariance matrix and the investor’s aversion to VaR, a confidence level (95%, 97.5%, 99%) has to be set in order to measure downside risk of the portfolio. The empirical analyses are provided using ten risky assets. The results indicate that the new approach generally performs better than the standard mean-variance model in terms of the optimal portfolios’ return as well as the expected risk premium per unit of risk. The empirical findings are persistent over the entire sample period (1985-2000) and robust across alternative investment horizons.

The paper is organized as follows. Section II constructs a mean-VaR efficient frontier. Section III introduces an optimal portfolio selection model in a value-at-risk framework. Section IV describes the data and presents the empirical results. Section V concludes the paper.

II. VaR-Efficient Portfolio

The importance of measuring the risk of a portfolio of financial assets has long been recognized by academics and practitioners. In recent years, the growth of trading activity and instances of financial market instability have prompted new studies underscoring the need for market participants to develop reliable risk measurement techniques. One technique advanced in the literature involves the use of Value-at-Risk models that determine how much the value of a portfolio could decline over a given period of time with a given probability as a result of changes in market prices or rates. VaR for a portfolio is an estimate of a specified percentile of the probability distribution of the portfolio’s value change over a given holding period. The specified percentile is usually in the lower tail of the empirical distribution, and in practice, VaR estimates are calculated from the 95th to 99th percentiles.

VaR is a measure of market risk for a portfolio of financial assets, and determines how much the value of a portfolio could decline over a given time horizon $\Delta$ with a given loss probability $\alpha$. Analytically, it can be formulated as follows:

$$\Phi_f[V_{t+\Delta}(q) - V_t(q) + VaR_f(q, \Delta, \alpha) < 0] = \alpha$$

(1)
where we consider $n$ financial assets whose prices at time $t$ are denoted by $P_{i,t}$, $i = 1, \ldots, n$. Then the value of a portfolio at time $t$ is written as $V_t(q) = \sum_{i=1}^{n} q_i P_{i,t}$, where $q_i$ denotes the number of shares invested in asset $i$. If the portfolio structure is held fixed between the current date $t$ and the future date $t+\Delta$, the change in the market value is given by $V_{t+\Delta}(q) - V_t(q) = \sum_{i=1}^{n} q_i (P_{i,t+\Delta} - P_{i,t})$. $\Phi_t$ is the conditional distribution of future asset prices given the information available at time $t$, and $VaR_t(q, \alpha, \Delta)$ is the VaR value of the portfolio with a time horizon $\Delta$. The confidence level $(1-\alpha)$ is typically chosen to be at least 95% and often as high as 99% or more depending on the time horizon. Hence, the VaR is the reserve amount such that the global position (portfolio plus reserve) only suffers a loss for a given small probability $\alpha$ over a fixed period of time.

It is possible to express the VaR measure in terms of return of the portfolio instead of portfolio value. Analytically, it can be formulated as follows:

$$\Phi_t[R_{t+\Delta} < -VaR_R(w, \alpha, \Delta)] = \alpha$$

where $R_{t+\Delta} = \sum_{i=1}^{n} w_i (\ln P_{t+\Delta} - \ln P_{t}) = \sum_{i=1}^{n} w_i R_{i,t+\Delta}$ is the portfolio return at time $t+\Delta$, $R_{i,t+\Delta}$ is the expected return on a risky asset $i$, and $w_i$ is the proportion of the portfolio held in asset $i$: $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$. $VaR_R(w, \alpha, \Delta)$ is the VaR value of portfolio returns $R_{t+\Delta}$ with a time horizon $\Delta$.

Clearly, VaR is simply a specific percentile of a portfolio’s potential loss distribution over a holding period. Assuming $R_{t+\Delta} \sim f_r$, where $f_r$ is a general return distribution, the VaR for time $t+\Delta$, estimated using a model indexed by $m$, conditional on the information available at time $t$ and denoted $VaR_m(\alpha, \Delta)$ is the solution to:

$$\int_{-\infty}^{VaR_m(\alpha, \Delta)} f_{m,t+\Delta}(x)dx = \alpha$$

Portfolio selection is generally based on a trade-off between expected return and risk, and requires a choice for the risk measure to be implemented. Usually, the risk is evaluated by the conditional second-order moment, i.e., conditional variance or volatility. This leads to the determination of the mean-variance efficient portfolio introduced by Markowitz (1952). It can also be based on a safety-first criterion (probability of failure), initially proposed by Roy (1952) and then implemented by Levy and Sarnat (1972) and Arzac and Bawa (1977). In this section, we use VaR of the portfolio (not the variance of the portfolio) as a measure of portfolio risk.
We define a VaR-efficient portfolio as a portfolio with allocation solving the constrained optimization problem:

\[
\begin{align*}
\max_w & \quad E_t(R_{t+\Delta}) \\
\text{s.t.} & \quad \text{VaR}_t(w, \alpha, \Delta) \leq \text{VaR}_b
\end{align*}
\]

where the VaR-efficient allocation depends on the loss probability level \(\alpha\), on the benchmark VaR level \(\text{VaR}_b\) limiting the authorized risk (in the context of capital adequacy requirement of the Basle Committee on Banking Supervision, usually one third or one quarter of the budget allocated to trading activities), and on the initial budget allocated at time \(t\) among \(n\) financial assets. The VaR-efficient allocation \(w^*\) solves the first-order conditions:

\[
\begin{align*}
E_t(\ln P_{t+\Delta} - \ln P_t) &= -\lambda^* \frac{\partial \text{VaR}_t}{\partial w}(w^*_t, \alpha, \Delta) \\
\text{VaR}_t(w^*_t, \alpha, \Delta) &= \text{VaR}_b
\end{align*}
\]

where \(\lambda^*\) is a Lagrange multiplier. By changing the \(\text{VaR}_b\) level in equation (4), the expected return of the portfolio is maximized subject to a VaR constraint, which yields the efficient portfolio frontier that consists of maximum expected return for a given VaR.

The VaR-efficient portfolio can also be obtained by minimizing the value at risk subject to an expected return constraint:

\[
\begin{align*}
\min_w & \quad \text{VaR}_t(w, \alpha, \Delta) \\
\text{s.t.} & \quad E_t(R_{t+\Delta}) = \bar{R}_b
\end{align*}
\]

where the portfolio’s expected return is restricted at \(\bar{R}_b\). For different values of \(\bar{R}_b\), value at risk of the portfolio is minimized subject to a return constraint to construct the mean-VaR efficient frontier. It includes only the combinations that yield minimum VaR for each expected return.

In this setup, given the distribution of individual stock returns (i.e., means and variance-covariance matrix), one can maximize the expected return of the portfolio subject to a VaR constraint or minimize VaR subject to an expected return constraint in order to find the set of mean-VaR efficient portfolios. We now assume that based on the mean-VaR efficient frontier, the optimal risky portfolio is determined and a given budget is allocated between the optimal risky portfolio and a risk-free asset. Then we let \(\theta\) be the proportion between the optimal risky portfolio and the risk-free asset. That is,

\[
R_{p,t+\Delta} = \theta \sum_{i=1}^{n} w^*_i R_{i,t+\Delta} + (1-\theta)r
\]
is a portfolio that leads to expected return \( E_t(R_{p,t+\Delta}) \) for this \( \theta \), \( r \) is the risk-free interest rate, and \( w_i \) is the proportion of security \( i \) in the optimal risky portfolio. The portfolio mean and VaR are given by:

\[
E_t(R_{p,t+\Delta}) = \theta \sum_{i=1}^n w_i E_i(R_{i,t+\Delta}) + (1-\theta)r
\]

\[
VaR_{p,t} = \theta \text{VaR}_{p,t}(w^*, \alpha)
\]

Substituting \( \theta \) from eq. (9) into eq. (8) yields the relationship between the portfolio’s expected return, \( E_t(R_{p,t+\Delta}) \), and its minimal VaR:

\[
E_t(R_{p,t+\Delta}) = r + \left( \frac{E^*_p - r}{\text{VaR}^*_p} \right) \text{VaR}_p
\]

where \( E^*_p = \sum_{i=1}^n w_i E_i(R_{i,t+\Delta}) \) and \( \text{VaR}^*_p = \text{VaR}^*_p(w^*, \alpha) \) are the expected return and VaR of the optimal risky portfolio.

Equation (10) indicates that the portfolio’s expected return \( E_t(R_{p,t+\Delta}) \) is positively related to its VaR, and describes all available portfolios created by mixing \( r \) and \( E^*_p \). Adding the risk-free asset, the investor can mix \( r \) and \( E^*_p \), creating an indefinitely large number of portfolios all lying on the straight line given in eq. (10). Investors can move along this line by varying the proportion of the risk-free asset according to their preferences. The risk-free asset allows investors a range of possible portfolios with different VaR and expected return characteristics that allow them to reach higher indifference curves. The slope coefficient in eq. (10) can be viewed as a Sharpe (1964) ratio in a mean-VaR framework.

### III. A Portfolio Selection Model

The modern portfolio theory aims to allocate financial assets by maximizing the expected value of some utility function or maximizing the expected risk premium per unit of risk. For a utility function \( U(W_t) \) depending on final wealth \( W_t \) and gross returns \( \tilde{R}_i \) for assets \( i = 1, 2, \ldots, n \), an investor’s optimal portfolio is determined by the standard asset allocation problem:

\[
\max_w \quad E(U) = E \left[ U(W_0 \sum_{i=1}^n w_i \tilde{R}_i) \right]
\]

\[
\text{s.t.} \quad w_i \geq 0, \quad \sum_{i=1}^n w_i = 1
\]
where \( E(U) \) is the investor's expected utility of wealth, \( W_0 \) is the investor's initial wealth, \( w_i \) are the portfolio weights that sum to one, and the asset returns are assumed to follow a multivariate normal distribution.

Following Freund (1956), Chopra and Ziemba (1993) and others assume a negative exponential utility function, \( U(W) = 1 - e^{-\gamma W} \), where \( \gamma \) indicates the investor's aversion to risk; the larger the value of \( \gamma \), the more conservative the investor. Assuming that the returns are normally distributed, Freund (1956) shows that the maximization of expected utility
\[
E(U) = \int_{-\infty}^{\infty} (1 - e^{-\gamma W}) e^{-(W-\mu)^2/2\sigma^2} dW
\]
is accomplished if we maximize the function:
\[
E(U) = \mu - \frac{\gamma}{2} \sigma^2
\]
where \( \mu \) and \( \sigma^2 \) are the expected return and variance of the portfolio, and \( \gamma/2 \) can be viewed as a measure of the investor’s risk tolerance. In other words, the expected utility maximization is equivalent to the mean-variance optimization problem:
\[
\max_{w} E(U) = \sum_{i=1}^{n} w_i E(\bar{R}_i) - \Psi \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E(\sigma_{ij}) \tag{14}
\]
s.t. \( w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \)
where \( \Psi \) is the investor’s risk tolerance, \( E(\bar{R}_i) \) is the expected return for asset \( i \), \( E(\sigma_{ij}) \) is the expected variance-covariance matrix; \( \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \), where \( \sigma_i \) and \( \sigma_j \) are the standard deviation of returns on asset \( i \) and \( j \), \( \rho_{ij} \) is the correlation between the returns on asset \( i \) and \( j \).

As shown in Levy and Markowitz (1979), the practice of using mean-variance analysis can also be justified based on the second-order Taylor-series expansion. The approximation of the utility function around the population mean \( \mu = E(\bar{R}_i) \) is:
\[
E[U(\bar{R}_i)] \approx U(\mu) + U'(\mu) E[(\bar{R}_i - \mu)] + 0.5 U''(\mu) E[(\bar{R}_i - \mu)^2] = U(\mu) + 0.5 U''(\mu)V \tag{15}
\]
where \( V = E[(\bar{R}_i - \mu)^2] \) corresponds to the population variance. Equation (15) indicates that the expected utility can be approximated with the sample mean and variance. A number of researchers have asserted that the right choice of mean-variance efficient portfolio will give precisely optimum expected utility if and only if all distributions are normal or investors have quadratic utility function. Taking this
utility function, it can be demonstrated that mean-variance decision making process leads to optimal choices. Nevertheless, the quadratic utility function has some properties that are hardly satisfied.

The mean-variance analysis developed by Markowitz critically relies on two assumptions: either the investors have quadratic utility or the asset returns are jointly normally distributed. Both assumptions are not required, just one or the other: (1) If an investor has quadratic preferences, she cares only about the mean and variance of returns; and the skewness and kurtosis of returns have no effect on expected utility, i.e., she will not care, for example, about extreme losses. Quadratic utility has been shown to be inconsistent with observed human choice behavior with respect to risk. (2) Mean-variance optimization can be justified if the asset returns are jointly normally distributed since the mean and variance will completely describe the distribution. The normal distribution is symmetric, thus its skewness or third moment is zero. The kurtosis or fourth moment of a normal distribution has a value equal to three. Unfortunately, the return distribution shows high peaks, fat tails and more outliers on the left or right tail. In other words, the empirical distribution of stock returns is typically skewed to the left (or right) and leptokurtic, meaning that extreme events occur much more frequently than predicted by the normal distribution. Since there is substantial evidence that the mean-variance approach is not an appropriate optimization technique, we introduce a framework that substitutes the variance with VaR.

The portfolio choice of a safety-first investor is to maximize expected return subject to a downside risk constraint. Roy’s (1952) and Arzac and Bawa’s (1977) safety-first investor uses Value-at-Risk as the measure for downside risk. We believe optimal portfolio selection under limited downside risk to be a practical problem. Even if agents are endowed with standard concave utility functions such that to a first order approximation they would be mean-variance optimizers, practical circumstances often impose constraints that elicit asymmetric treatment of upside potential and downside risk. Regulatory concerns require commercial banks to report a single number, the so-called VaR, which gives the expected loss on their trading portfolio if the lowest 1% quantile return would materialize. Capital adequacy is judged on the basis of the size of this expected loss. Likewise, pension funds are often required by law to structure their investment portfolio such that the risk of underfunding is kept low, e.g., equity investment may be capped.

In this paper, we assume that VaR provides good predictions of catastrophic market risks during ordinary and extraordinary periods. Thus, in the same way that Levy and Markowitz (1979) assume that an individual’s expected utility in portfolio selection can be closely approximated by a function of only the mean and variance, we assume that the expected utility in optimal asset allocation can be closely approximated by a function of the mean, VaR, and the risk tolerance. Investors are
assumed to use optimal portfolio selection under limited downside risk as an alternative to traditional mean-variance approach.

To show how the investor can obtain a mean-VaR asset allocation model, we need to define VaR of the portfolio $VaR_p(w, \alpha)$ explicitly. Since the asset returns are skewed and fat-tailed we cannot use a VaR formula that assumes a normal distribution. We estimate value at risk using the Cornish-Fisher (1937) expansion that adjusts the traditional VaR with the skewness and kurtosis of the empirical distribution:

$$VaR_p(w, \alpha) = R_p(w) - \Omega(\alpha) \times \sigma_p(w)$$

(16)

where $R_p(w) = \sum_{i=1}^n w_i E(R_i)$ is the expected return of the portfolio, $\sigma_p(w) = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j E(\sigma_{ij})}$ is the expected standard deviation of the portfolio, and $\Omega(\alpha)$ is the critical value based on the loss probability level, skewness, and kurtosis of the empirical distribution:

$$\Omega(\alpha) = z(\alpha) + \frac{1}{6} (z(\alpha)^2 - 1) S + \frac{1}{24} (z(\alpha)^3 - 3z(\alpha)) K - \frac{1}{36} (2z(\alpha)^3 - 5z(\alpha)) S^2$$

(17)

with $z(\alpha)$ is the critical value from the normal distribution for probability $(1-\alpha)$, $S$ is the skewness, and $K$ is the excess kurtosis. Equation (17) indicates that the Cornish-Fisher expansion allows us to compute VaR for distribution with asymmetry and leptokurtosis. Note that if the distribution is normal, $S$ and $K$ are equal to zero, which makes $\Omega(\alpha)$ be equal to $z(\alpha)$.

This paper develops an optimal portfolio selection model which maximizes the expected utility of final wealth defined as a function of expected return and downside risk of the portfolio. In our approach, downside risk is specified in terms of portfolio VaR rather than standard deviation alone so that additional risk resulting from any non-normality (such as skewness and excess kurtosis) can be used to estimate the portfolio VaR. In the mean-VaR asset allocation model, the investor’s optimal portfolio is the solution to:

$$\max_w E(U) = R_p(w) - \Gamma VaR_p(w, \alpha)$$

(18)

s.t. $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$

where $\Gamma$ is the investor’s aversion to Value-at-Risk or the risk tolerance in a mean-VaR framework. The mean-VaR approach aims to allocate financial assets by maximizing the expected value of a utility function approximated by the expected return $R_p(w)$ and VaR of the portfolio $VaR_p(w, \alpha)$ as well as the investor’s aversion to VaR denoted by $\Gamma$. In the mean-variance framework, market risk of the
portfolio is measured by the possible variation of expected portfolio returns, and investors are assumed to weight the probability of negative returns equally against positive returns. Whereas, using VaR as the appropriate measure of portfolio risk, investors are allowed to treat gains and losses asymmetrically, and are allowed to measure the negative and positive tails of the return distribution separately.

As indicated by Champbell et al. (2001), the existing literature does not provide a methodology for estimating the investor’s aversion to market risk of the portfolio. In this paper, we introduce a new and robust procedure to estimating time-varying risk tolerance in the mean-variance and mean-VaR framework. Maximizing the expected utility in eq. (14) with respect to portfolio weights,

\[
\frac{\partial E(U)}{\partial w_i} = E(\hat{R}_i) - \Psi_i \left[ 2w_i E(\sigma_i^2) + 2 \sum_{j=1}^{n} w_j E(\sigma_{ij}) \right] = 0 \quad \text{for } i = 1, 2, \ldots, n
\]

gives the possible values of risk tolerance for the mean-variance asset allocation model:

\[
\Psi_i = \frac{E(\hat{R}_i)}{2w_i E(\sigma_i^2) + 2 \sum_{j=1 \atop j \neq i}^{n} w_j E(\sigma_{ij})}
\]

(19)

The degree of risk aversion in eq. (19) indicates that the investor’s risk tolerance is a function of the expected returns \( E(\hat{R}_i) \), variances \( E(\sigma_i^2) \), covariances \( E(\sigma_{ij}) \), and the set of portfolio weights. The expected returns and the variance-covariance matrix are obtained from the monthly data described above. The portfolio weights are calculated by maximizing the Sharpe ratio \( \frac{R_p(w) - r}{\sigma_p(w)} \) in a mean-variance framework. Alternatively, the weights are obtained from the mean-variance efficient frontier. As presented in Table 2, the estimated values of risk tolerance are not sensitive to the set of portfolio weights used in eq. (19). Since our empirical analyses are based on ten risky assets (\( n = 10 \)), expected utility maximization yields ten risk aversion parameters. We use the average values of \( \Psi_i \)'s (\( i = 1, 2, \ldots, 10 \)) in optimal portfolio selection.

A similar procedure is used to estimate the investor’s aversion to value-at-risk. Specifically, we maximize the expected utility in eq. (18) with respect to portfolio weights,
\[ \frac{\partial E(U)}{\partial w_i} = E(R_i) - \Gamma_i \left[ 0.5 \Omega(\alpha) \left( 2w_i E(\sigma_i^2) + 2 \sum_{j \neq i}^n w_j E(\sigma_j) \right)^{0.5} - E(R_i) \right] = 0 \quad \text{for } i = 1, 2, \ldots, n = 10 \]

and obtain the possible values of risk tolerance for the mean-VaR portfolio selection model:

\[ \Gamma_i = \frac{E(R_i)}{0.5 \Omega(\alpha) \left( 2w_i E(\sigma_i^2) + 2 \sum_{j \neq i}^n w_j E(\sigma_j) \right)^{0.5} - E(R_i)} \]

where \( \Omega(\alpha) \) is the critical value depending on the mean, variance, skewness, and kurtosis of the empirical distribution. As mentioned earlier, the expected returns and the variance-covariance matrix are obtained from the monthly data on ten DJIA securities. The portfolio weights are calculated by maximizing the Sharpe ratio in a mean-VaR framework: \([R_p(w) - r]/VaR_p(w, \alpha)\). Table 2 shows that when the weights are obtained from the mean-VaR efficient frontier, the estimated values of \( \Gamma_i \) are found to be very similar to the original estimates based on the Sharpe ratio. Since ten risky assets are used in our empirical analyses, expected utility maximization yields ten \( \Gamma_i \) parameters for the investor’s aversion to VaR. We use their average values in optimal asset allocation.

We should note that the same optimization problem is not used to find the risk aversion parameters, \( \Psi \) and \( \Gamma \), in the mean-variance and mean-VaR framework. In other words, the estimation procedure is not circular. The first order conditions are used to define \( \Psi \) and \( \Gamma \) as a function of the known parameters – the means, variances, and covariances – and the unknowns – the portfolio weights. As discussed above and shown in Table 2, the set of portfolio weights are obtained from the Sharpe ratio maximization and from five different points on the mean-variance or mean-VaR efficient frontier. The first set of portfolio weights is obtained by minimizing the standard deviation (or VaR) of the portfolio. The second set is determined by maximizing the expected portfolio returns. These two sets yield the two end points (minimum risk and maximum return) of the mean-variance (or mean-VaR) efficient frontier. The third set includes the portfolio weights that yield a point on the efficient frontier.
⅛ way from maximum return and ⅜ way from minimum risk. The fourth set contains the portfolio weights that yield a risk-return combination that is half way from minimum risk and maximum return. The last set is comprised of the portfolio weights that correspond to a point on the efficient frontier ⅛ way from maximum return and ⅜ way from minimum risk. Table 2 provides strong evidence that the investor’s aversion to portfolio risk is not sensitive to the set of portfolio weights.

IV. Data and Empirical Results

The data consist of monthly observations from January 1980 through December 2000 on ten Dow Jones Industrial Average (DJIA) securities. The total returns on DJIA securities are obtained from the Center for Research in Securities Prices (CRSP) database. The summary statistics of monthly returns on DJIA securities are presented in Table 1. The average returns are in the range of 1.40% to 2.16% per month. The unconditional standard deviations range from 5.81% to 8.83% per month. The correlation coefficients are greater than 0.16 and less than 0.66.

This section analyzes an investor’s dynamic asset allocation (DAA) strategy based on a risk-return trade-off. In a DAA framework, using five years of monthly returns (from January 1980 through December 1984), a dynamic asset allocator estimates the one-period-ahead returns, variances, and covariances for ten risky assets as well as the risk tolerance, and re-balances the portfolio accordingly to determine the optimum asset mix for the next month. Then he invests 100% of the portfolio in the optimum asset mix and hold for one month. With new information and market conditions, the investor will go through the same forecasting and re-balancing process at the beginning of each month from January 1985 through December 2000, yielding a total of 192 observations for the optimal portfolios’ return and risk. The relative performance of the mean-variance and mean-VaR portfolio selection models is then compared in terms of the optimal portfolios’ return as well as the expected risk premium per unit of risk (or the risk-adjusted return).

To determine the optimum asset mix in a DAA framework, we maximize the expected utility of final wealth using the means, variance-covariance matrix and the investor’s aversion to portfolio risk as inputs. The risk tolerance reflects the investor’s desired trade-off between extra return and extra risk. The smaller the risk tolerance, the more risk an investor is willing to take for a little extra return. Chopra and Ziemba (1993) mention that under fairly general input assumptions, a risk tolerance of 0.04 describes the typical portfolio allocations of large U.S. pension funds and other institutional investors. They also point out that risk tolerances of 0.02 and 0.08 in a mean-variance framework characterize extremely aggressive and conservative investors, respectively. However, Chopra and Ziemba do not
provide any methods to determine the investor’s risk aversion parameter. We introduce a robust procedure to estimate risk tolerance in mean-variance and mean-VaR framework. As shown in equations (16)-(17), the investor’s aversion to risk depends on the expected returns, variances, covariances, and the set of portfolio weights. We test the sensitivity of risk tolerance to the aforementioned factors affecting the investor’s aversion to portfolio risk.

The set of portfolio weights used in the calculation of risk tolerance is obtained by maximizing the Sharpe ratio or from the mean-variance or mean-VaR efficient frontier. Specifically, we maximize $[R_p(w) - r]/\sigma_p(w)$ in mean-variance and $[R_p(w) - r]/VaR_p(w, \alpha)$ in mean-VaR framework using five years of monthly returns to find the investor’s aversion to portfolio risk for the next month. The total sample of 252 monthly observations for ten risky assets from January 1980 to December 2000 is split into two parts: the first 60 observations from January 1980 to December 1984 are used for estimation of risk aversion parameters for January 1985. The sample is then rolled forward by removing the first observation of the sample and adding one to the end, and another set of parameters is obtained. This recursive estimation procedure is repeated until the risk aversion parameters for observation 252 (or for December 2000) have been obtained using the data available at time 251. This yields a total of 192 observations for the time-varying risk tolerance in mean-variance and mean-VaR framework.

Figure 1 plots the investor’s aversion to portfolio risk measured by the variance and value-at-risk at the 1%, 2.5%, and 5% loss probability levels. As expected, the risk tolerance in mean-variance framework is found to be much lower but highly correlated with the risk tolerance in the mean-VaR framework. Specifically, the investor’s aversion to value-at-risk is at least four times greater than the risk tolerance for the mean-variance model. To illustrate clearly the differences in magnitudes of risk tolerances and the correlations between them, we multiply the risk tolerance of the mean-variance model by four and then plot it in Figure 1. The correlations between time-varying risk aversion parameters are found to be greater than 0.96. Alternatively, we estimate the time-varying risk tolerance based on five sets of portfolio weights obtained from the mean-variance and mean-VaR efficient frontiers. The results are found to be almost the same as in Figure 1.13

The first set of portfolio weights is obtained by minimizing the standard deviation (and VaR) of the portfolio. The second set is determined by maximizing the expected portfolio returns. These two sets yield the two end points (minimum risk and maximum return) of the mean-variance (and mean-VaR) efficient frontier. The third set includes the portfolio weights that yield a point on the efficient frontier $1/4$ way from maximum return and $3/4$ way from minimum risk. The fourth set contains the portfolio weights that yield a risk-return combination that is half way from minimum risk and
maximum return. The last set is comprised of the portfolio weights that correspond to a point on the efficient frontier \(\frac{3}{4}\) way from maximum return and \(\frac{1}{4}\) way from minimum risk. We estimate the investor’s risk tolerance using the portfolio weights obtained from the Sharpe ratio maximization and from five different points on the efficient frontier. Table 2 provides strong evidence based on the entire sample (January 1980-December 2000) estimates that the investor’s aversion to portfolio risk is not sensitive to the set of portfolio weights. Specifically, the risk tolerance for the mean-variance model is found to be in the range of 0.0335 to 0.0391. The investor’s aversion to VaR ranges from 0.1715 to 0.1968 for the 1% VaR, 0.2108 to 0.2443 for the 2.5% VaR, and 0.2609 to 0.3066 for 5% VaR.

As expected, portfolio risk in mean-variance and mean-VaR framework is negatively related to the risk tolerance. That is, the greater risk the investor is willing to take for a little extra return, the greater the standard deviation of the portfolio. To measure the magnitude of this relation between the investors’ aversion to portfolio risk and the standard deviation (or market risk) of the portfolio, we use 11 sets of portfolio weights obtained from the mean-variance efficient frontier and calculate the corresponding risk tolerance and portfolio risk. The correlation between risk tolerance and standard deviation of the portfolio turns out to be –0.9480. As shown in Figure 2, portfolio risk (measured by the standard deviation) declines as the investor becomes more conservative (or more risk averse).

In order to evaluate the relative performance of the mean-variance and mean-VaR portfolio selection models, we use the cumulative cash values of $100 investment, the compounded annual return (or the annualized geometric mean return), and the expected risk premium per unit of risk (or the risk-adjusted return). Before presenting the results on the aforementioned performance measures, we display the expected returns and risks of the portfolios for alternative investment horizons. Figure 3 indicates that the expected return of the mean-variance portfolio is slightly greater than that of the mean-VaR portfolio, and this result is robust according to the length of investment time horizon (1-month, 3-month, 6-month, and 1-year). We should note that since the expected returns of the mean-VaR portfolios turn out to be almost the same for the 1%, 2.5%, and 5% VaRs, we choose to present the results for the mean-1% VaR portfolio only. As expected, the standard deviation of the mean-VaR portfolio is less than that of the mean-variance portfolio, and as shown in Figure 4, this result is robust across different investment horizons. Panels A and B of Table 3 present the average expected returns and standard deviations of the mean-variance and mean-VaR portfolios for an investment horizon of 1 month, 3 months, 6 months, and 1 year.

To determine whether the mean-VaR portfolio performs better or worse than the mean-variance portfolio in terms of the expected risk premium per unit of risk, we calculate the risk-adjusted
returns at the beginning of each month from January 1985 through December 2000. More specifically, we calculate the ratio of excess return to standard deviation of the portfolio. When computing the excess return (expected return – risk-free rate) per unit of risk, we identify the risk-free rate with the time-series of 3-month Treasury bill rates. Figure 5 indicates that the mean-VaR approach performs slightly better than the mean-variance model in terms of the optimal portfolios’ excess return per unit of risk (or the risk-adjusted return). This finding is persistent over the entire sample period (1985-2000) and robust according to the length of investment horizon. Panel C of Table 3 presents the average risk-adjusted returns on the mean-variance and mean-VaR portfolios for alternative investment horizons.

Panel A of Table 4 shows that for the period of January 1985 to December 2000, the mean-variance asset allocation model provides an average compounded annual return of 15.95%, whereas the mean-VaR approach generates an average compounded annual return of 17.42% for the 1% VaR, 17.43% for the 2.5% VaR, and 17.44% for the 5% VaR. As indicated by Panel B of Table 4, the annualized geometric mean return on the mean-VaR portfolio is greater than that on the mean-variance portfolio by almost 1.50% for a 1-month investment horizon, 1.70% for a 3-month investment horizon, 1.67% for a 6-month investment horizon, and 1.06% for a 1-year investment horizon. According to cumulative cash values displayed in Table 4, if one had invested $100 in the mean-variance portfolio he would have received $1,261.86 within 16 years. During the same period, if he had invested $100 in the mean-VaR portfolio he would have received $1,592.04 from the mean-1% VaR portfolio, $1,594.39 from the mean-2.5% VaR portfolio, and $1,597.58 from the mean-5% VaR portfolio. Figure 6 indicates that the mean-VaR approach performs better than the mean-variance model in terms of the cumulative cash values of $100 investment and the compounded annual return (or the annualized geometric mean return). This finding is persistent over the entire sample period (1985-2000) and robust according to the length of investment time horizon.

V. Conclusions

This paper develops a dynamic asset allocation model in a value-at-risk framework where the market risk of the portfolio depends on the portfolio’s potential loss function. Introducing VaR as an alternative measure of portfolio risk has enabled us to analyze the risk-return trade-off for various associated confidence levels. Since in the new approach the riskiness of an asset increases with the choice of loss probability levels associated with downside risk measure, the penalty for risk becomes a function of VaR and the individual’s risk aversion level. The mean-VaR approach introduced here allocates financial assets by maximizing the expected value of some utility function approximated by the
expected return and VaR of the portfolio as well as the investor’s aversion to value-at-risk. In other words, the portfolio selection problem is to maximize expected return while minimizing downside risk of the portfolio for alternative loss probability levels. The relative performance of the mean-variance and mean-VaR portfolio selection models is compared in a dynamic asset allocation framework. The empirical findings indicate that the new approach generally performs better than the mean-variance model in terms of the optimal portfolios’ return as well as the expected risk premium per unit of risk. The results are persistent for the period from 1985 to 2000 and robust across alternative investment horizons. In addition, the paper introduces a new robust procedure to estimating the investor’s aversion to portfolio risk in mean-variance and mean-VaR framework. The results indicate that the investor’s risk tolerance is time-varying and closely related to the standard deviation (or market risk) of the portfolio.
References


Table 1
Summary Statistics
The data consist of monthly observations from January 1980 through December 2000 on ten Dow Jones Industrial Average (DJIA) securities: Aluminum Co. of America (Alcoa), American Express Co. (Amex), Boeing Co. (Boeing), Chevron Co. (Chev.), Coca Cola Co. (Coke), E.I. Du Pont De Nemours & Co. (Du Pont), Minnesota Mining and Manufacturing Co. (MMM), Procter & Gamble Co. (P&G), Sears, Roebuck & Co. (Sears), United Technologies Co. (U Tech). The total returns on DJIA securities are obtained from the Center for Research in Securities Prices (CRSP) database. The unconditional means, standard deviations, and correlations of monthly returns are presented below.

<table>
<thead>
<tr>
<th></th>
<th>Alcoa</th>
<th>Amex</th>
<th>Boeing</th>
<th>Chev.</th>
<th>Coke</th>
<th>Du Pont</th>
<th>MMM</th>
<th>P&amp;G</th>
<th>Sears</th>
<th>U Tech</th>
</tr>
</thead>
<tbody>
<tr>
<td>Means</td>
<td>1.5476</td>
<td>1.8795</td>
<td>1.6152</td>
<td>1.3594</td>
<td>1.9625</td>
<td>1.3661</td>
<td>1.3803</td>
<td>1.5992</td>
<td>1.3453</td>
<td>1.6461</td>
</tr>
</tbody>
</table>

Correlations

<table>
<thead>
<tr>
<th></th>
<th>Alcoa</th>
<th>Amex</th>
<th>Boeing</th>
<th>Chev.</th>
<th>Coke</th>
<th>Du Pont</th>
<th>MMM</th>
<th>P&amp;G</th>
<th>Sears</th>
<th>U Tech</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amex</td>
<td>0.3437</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boeing</td>
<td>0.2958</td>
<td>0.4463</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chev.</td>
<td>0.1916</td>
<td>0.2639</td>
<td>0.2131</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coke</td>
<td>0.1387</td>
<td>0.4179</td>
<td>0.4009</td>
<td>0.0726</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Du Pont</td>
<td>0.4884</td>
<td>0.4721</td>
<td>0.3881</td>
<td>0.3849</td>
<td>0.2982</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MMM</td>
<td>0.4974</td>
<td>0.3663</td>
<td>0.4401</td>
<td>0.2363</td>
<td>0.3471</td>
<td>0.5563</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P&amp;G</td>
<td>0.1346</td>
<td>0.3839</td>
<td>0.3173</td>
<td>0.1121</td>
<td>0.5316</td>
<td>0.3740</td>
<td>0.3930</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sears</td>
<td>0.2263</td>
<td>0.4950</td>
<td>0.3886</td>
<td>0.2334</td>
<td>0.3751</td>
<td>0.4568</td>
<td>0.3949</td>
<td>0.3343</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>U Tech</td>
<td>0.4648</td>
<td>0.5032</td>
<td>0.5607</td>
<td>0.3867</td>
<td>0.3540</td>
<td>0.5069</td>
<td>0.5030</td>
<td>0.3405</td>
<td>0.4515</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Table 2
Sensitivity of Risk Tolerance to the Expected Returns, Variance-Covariance Matrix, and the Set of Portfolio Weights
(January 1980–December 2000)

This table displays the investor’s aversion to portfolio risk measured by the variance or value-at-risk of the portfolio based on the entire-sample (January 1980–December 2000) estimates. The investor’s risk tolerance is estimated using the portfolio weights obtained from the Sharpe ratio maximization and from five different points on the mean-variance or mean-VaR efficient frontier. The first set of portfolio weights is obtained by minimizing the standard deviation (or VaR) of the portfolio. The second set is determined by maximizing the expected portfolio returns. These two sets yield the two end points (minimum risk and maximum return) of the mean-variance (or mean-VaR) efficient frontier. The third set includes the portfolio weights that yield a point on the efficient frontier ¼ way from maximum return and ¾ way from minimum risk. The fourth set contains the portfolio weights that yield a risk-return combination that is half way from minimum risk and maximum return. The last set is comprised of the portfolio weights that correspond to a point on the efficient frontier ¾ way from maximum return and ¼ way from minimum risk.

<table>
<thead>
<tr>
<th>Average Risk Tolerance</th>
<th>Sharpe Ratio Maximization</th>
<th>Efficient Frontier Minimum Risk</th>
<th>Efficient Frontier ¼ Max Return</th>
<th>Efficient Frontier ½ Max Return</th>
<th>Efficient Frontier ¾ Max Return</th>
<th>Efficient Frontier Maximum Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>0.0378</td>
<td>0.0391</td>
<td>0.0385</td>
<td>0.0371</td>
<td>0.0348</td>
<td>0.0335</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>0.1715</td>
<td>0.1720</td>
<td>0.1724</td>
<td>0.1743</td>
<td>0.1796</td>
<td>0.1968</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>0.2108</td>
<td>0.2113</td>
<td>0.2122</td>
<td>0.2147</td>
<td>0.2217</td>
<td>0.2443</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>0.2609</td>
<td>0.2614</td>
<td>0.2630</td>
<td>0.2665</td>
<td>0.2761</td>
<td>0.3066</td>
</tr>
</tbody>
</table>
Table 3
Annualized Average Expected Return, Standard Deviation, and Risk-Adjusted Returns of the Mean-Variance and Mean-VaR Portfolios Over the 1985-2000 Period

This table presents the annualized average expected return, standard deviation, and expected risk premium per unit of risk (risk-adjusted return) of the mean-variance and mean-VaR portfolios for an investment time horizon (or holding period) of 1 month, 3 months, 6 months, and 1 year.

Panel A
Annualized Average Expected Return of the Mean-Variance and Mean-VaR Portfolios

<table>
<thead>
<tr>
<th>Portfolio Selection Model</th>
<th>Expected Return 1-month</th>
<th>Expected Return 3-month</th>
<th>Expected Return 6-month</th>
<th>Expected Return 1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>23.6491 %</td>
<td>23.6159 %</td>
<td>23.6624 %</td>
<td>23.3471 %</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>22.6091 %</td>
<td>22.6068 %</td>
<td>22.6134 %</td>
<td>22.3691 %</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>22.6023 %</td>
<td>22.6004 %</td>
<td>22.6068 %</td>
<td>22.3618 %</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>22.5930 %</td>
<td>22.5914 %</td>
<td>22.5974 %</td>
<td>22.3522 %</td>
</tr>
</tbody>
</table>

Panel B
Annualized Average Standard Deviation of the Mean-Variance and Mean-VaR Portfolios

<table>
<thead>
<tr>
<th>Portfolio Selection Model</th>
<th>Standard Deviation 1-month</th>
<th>Standard Deviation 3-month</th>
<th>Standard Deviation 6-month</th>
<th>Standard Deviation 1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>16.1055 %</td>
<td>16.0282 %</td>
<td>16.0363 %</td>
<td>15.8607 %</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>15.0960 %</td>
<td>15.0739 %</td>
<td>15.0607 %</td>
<td>14.9565 %</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>15.0908 %</td>
<td>15.0691 %</td>
<td>15.0556 %</td>
<td>14.9512 %</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>15.0838 %</td>
<td>15.0626 %</td>
<td>15.0488 %</td>
<td>14.9443 %</td>
</tr>
</tbody>
</table>

Panel C
Annualized Average Risk-Adjusted Returns on the Mean-Variance and Mean-VaR Portfolios

<table>
<thead>
<tr>
<th>Portfolio Selection Model</th>
<th>Risk-Adjusted Return 1-month</th>
<th>Risk-Adjusted Return 3-month</th>
<th>Risk-Adjusted Return 6-month</th>
<th>Risk-Adjusted Return 1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>1.1741</td>
<td>1.1791</td>
<td>1.1807</td>
<td>1.1653</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>1.1828</td>
<td>1.1862</td>
<td>1.1902</td>
<td>1.1727</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>1.1827</td>
<td>1.1861</td>
<td>1.1901</td>
<td>1.1726</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>1.1826</td>
<td>1.1860</td>
<td>1.1900</td>
<td>1.1725</td>
</tr>
</tbody>
</table>
Table 4
The Relative Performance of the Mean-Variance and Mean-VaR Portfolio Selection Models (January 1985–December 2000)

This table presents the cumulative cash values of $100 investment and the compounded annual return (or the annualized geometric mean return) on the mean-variance and mean-VaR portfolio selection models for an investment time horizon of 1-month, 3 months, 6 months, and 1 year.

Panel A
Value of $100 Investment on Mean-Variance and Mean-VaR Portfolios in 16 Years

<table>
<thead>
<tr>
<th>Portfolio Selection Model</th>
<th>Value of $100 1-month</th>
<th>Value of $100 3-month</th>
<th>Value of $100 6-month</th>
<th>Value of $100 1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>$1,261.86</td>
<td>$1,415.82</td>
<td>$1,589.03</td>
<td>$1,701.24</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>$1,592.04</td>
<td>$1,834.87</td>
<td>$2,018.93</td>
<td>$1,956.11</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>$1,594.39</td>
<td>$1,836.99</td>
<td>$2,022.58</td>
<td>$1,957.76</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>$1,597.58</td>
<td>$1,840.43</td>
<td>$2,027.92</td>
<td>$1,959.98</td>
</tr>
</tbody>
</table>

Panel B
Annualized Geometric Mean Return on Mean-Variance and Mean-VaR Portfolios

<table>
<thead>
<tr>
<th>Portfolio Selection Model</th>
<th>Geometric Mean Return 1-month</th>
<th>Geometric Mean Return 3-month</th>
<th>Geometric Mean Return 6-month</th>
<th>Geometric Mean Return 1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>15.9499 %</td>
<td>16.9121 %</td>
<td>18.0547 %</td>
<td>19.3776 %</td>
</tr>
<tr>
<td>Mean-1% VaR</td>
<td>17.4228 %</td>
<td>18.6044 %</td>
<td>19.6924 %</td>
<td>20.4237 %</td>
</tr>
<tr>
<td>Mean-2.5% VaR</td>
<td>17.4321 %</td>
<td>18.6120 %</td>
<td>19.7048 %</td>
<td>20.4301 %</td>
</tr>
<tr>
<td>Mean-5% VaR</td>
<td>17.4448 %</td>
<td>18.6242 %</td>
<td>19.7229 %</td>
<td>20.4386 %</td>
</tr>
</tbody>
</table>
Figure 1: Time-Varying Risk Tolerance: Mean-Variance versus Mean-VaR

Figure 2: Risk Tolerance versus Standard Deviation of the Portfolio
Figure 3.A: Expected Return of the Mean-Variance and Mean-VaR Portfolios with Monthly Update

Figure 3.B: Expected Return of the Mean-Variance and Mean-VaR Portfolios with Quarterly Update

Figure 3.C: Expected Return of the Mean-Variance and Mean-VaR Portfolios with Annual Update
Figure 5.A: Expected Risk Premium Per Unit of Risk with Monthly Update

Figure 5.B: Expected Risk Premium Per Unit of Risk with Quarterly Update

Figure 5.C: Expected Risk Premium Per Unit of Risk with Annual Update
Figure 6.A: Value of $100 Investment on Mean-Variance and Mean-VaR Portfolios with Monthly Update

Figure 6.B: Value of $100 Investment on Mean-Variance and Mean-VaR Portfolios with Quarterly Update

Figure 6.C: Value of $100 Investment on Mean-Variance and Mean-VaR Portfolios with Annual Update
Endnotes

1 See Longin (2000) and Bali (2001).
2 Value at risk (VaR) is defined as the expected maximum loss over a given time interval at a given confidence level. For example, if the given period of time is one day and the given probability is 1%, the VaR measure would be an estimate of the decline in the portfolio value that could occur with a 1% probability over the next trading day. In other words, if the VaR measure is accurate, losses greater than the VaR measure should occur less than 1% of the time. For a comprehensive survey on VaR models, see Jorion (1997).
3 For a risk tolerance of 0.04, errors in means are about eleven times as important as errors in variances, and errors in variances are about twice as important as errors in covariances. At lower risk tolerances, errors in means are even more important relative to errors in variances and covariances. At higher risk tolerances, the relative impact of errors in means, variances, and covariances is closer. Although errors in means are more important than those in variances-covariances, the difference in importance diminishes with an increase in risk tolerance.
4 As will be discussed in Section III, the investor’s risk tolerance in a mean-VaR framework also depends on the confidence level chosen to measure downside risk.
5 VaR models aggregate the several components of price risk into a single quantitative measure of the potential for losses over a specified time horizon. These models are clearly appealing because they convey the market risk of the entire portfolio in one number. Moreover, VaR measures focus directly, and in dollar terms, on a major reason for assessing risk in the first place – a loss of portfolio value.
6 Freund (1956, p.255) indicates that this is accomplished by completing the square in the exponent, which produces a normal integral multiplied by $-e^{-(\gamma^2/2)}$. Since the integral becomes a constant, a maximization of $E(U)$ in equation (12) is equivalent to minimizing the exponent which is in turn accomplished by maximizing equation (13).
7 Leibowitz and Kogelman (1991) and Lucas and Klaassen (1998) also construct portfolios by maximizing expected return subject to a shortfall constraint, defined as a minimum return that should be gained over a given time horizon for a given confidence level.
8 For example, $z(\alpha)$ equals -2.33 (-1.96) [-1.65] for the 1% (2.5%) [5%] VaR.
9 At an earlier stage of the study, we generated an equally weighted index of ten risky assets considered in the paper and then found 1% of the right and left tails of the empirical distribution: $\Omega(\alpha = 1\%) = 2.8473$ for the right tail and $\Omega(\alpha = 1\%) = 2.6684$ for the left tail. The mean and standard deviation of the distribution of monthly returns are, respectively, 1.5701% and 4.9248%, and 1% of the right and left tails correspond to the monthly returns of 15.59% and -11.57%, respectively. The results indicate that asset returns exhibit skewness and significant excess kurtosis. The skewness and excess kurtosis statistics of monthly returns on equally-weighted index are, respectively, -0.5523 (-0.1543) and 7.0026 (0.3086) with standard errors in parentheses.
10 Harvey and Siddique (1999, 2000) and Bekaert et al. (1998) advocate the need to incorporate non-normalities (skewness and excess kurtosis) into the portfolio selection decision.
11 Chambell, Huisman, and Koedijk (2001, p.1792) state that “The degree of risk aversion is set according to the VaR limit; hence avoiding the limitations of expected utility theory as to the degree of risk aversion, which an investor is thought to exhibit.”
12 This is an extended version of the data set used by Chopra and Ziemba (1993).
13 To save space we do not display the time-varying risk tolerances estimated based on the set of portfolio weights obtained from the mean-variance and mean-VaR efficient frontiers. They are available upon request.
14 Figure 2 displays a scatter diagram for the investor’s aversion to portfolio risk and the standard deviation of the portfolio in a mean-variance framework. The same relation between the risk tolerance and the standard deviation of the portfolio is obtained for the mean-1%VaR, mean-2.5%VaR, and mean-5%VaR models. To preserve space we do not present the figures for the mean-VaR framework. They are available upon request.
15 The performance measures for the mean-variance and mean-VaR portfolios are found to be robust across different investment horizons. To preserve space, we do not present the results for the 6-month investment horizon. They are available upon request.
16 To keep the graphs readable, we do not show the expected returns of the mean-1%VaR, mean-2.5%VaR, and mean-5%VaR portfolios in the same figure.